Axiomatizing modal fixpoint logics

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(largely joint work with Enqvist, Seifan, Santocanale, Schröder, ... )
Modal Fixpoint Logics

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- **Modal fixpoint languages** extend basic modal logic with either
  - new fixpoint connectives such as \( \langle \ast \rangle, U, C, \ldots \) \( \leadsto \) LTL, CTL, PDL
  - explicit fixpoint operators \( \mu x, \nu x \leadsto \mu ML \)
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- **Motivation 1**: increase expressive power
  - e.g. enable specification of ongoing behaviour

- **Motivation 2**: generally nice computational properties

Combined: many applications in process theory, epistemic logic, etc.

Interesting mathematical theory:
- interesting mix of algebraic | coalgebraic features & combinatorics
- connections with theory of automata on infinite objects
- intuitive game-theoretical semantics
- interesting meta-logic
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General Program

Understand modal fixpoint logics by studying the interaction between

- combinatorial
- algebraic and
- coalgebraic

aspects

Here: consider axiomatization problem
Axiomatization of fixpoints

Least fixpoint $\mu p. \varphi$ should be axiomatized by
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Least fixpoint $\mu p. \varphi$ should be axiomatized by

- a least (pre-)fixpoint axiom:

$$\varphi(\mu p. \varphi) \vdash \mu p. \varphi$$

- Park's induction rule

$$\frac{\varphi(\psi) \vdash \varphi}{\mu p. \varphi \vdash \psi}$$

(Here $\alpha \vdash_{K} \beta$ abbreviates $\vdash_{K} \alpha \rightarrow \beta$)
Axiomatization results for modal fixpoint logics

- LTL: Gabbay et alii (1980)
- $\mu ML$ (aconjunctive fragment): Kozen (1983)
- CTL: Emerson & Halpern (1985)
- $\mu ML$: Walukiewicz (1993/2000)
- CTL*: Reynolds (2000)
- LTL/CTL uniformly: Lange & Stirling (2001)
- common knowledge logics: various
- ...
Axiomatization results for modal fixpoint logics

▶ LTL: Gabbay et alii (1980)
▶ PDL: Kozen & Parikh (1981)
▶ $\mu$ML (aconjunctive fragment): Kozen (1983)
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So what is the problem?
Questions (2015)

- How to prove completeness for new fixpoint logics?
- How to transfer known results to restricted frame classes?
- How to transfer known results to similar logics e.g. the monotone $\mu$-calculus?
- Does completeness transfer to fragments of $\mu$ML? (Ex: game logic)
- What about proof theory?
- ...
Axiomatization problem

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Compared to basic modal logic

- there are no sweeping general results such as Sahlqvist’s theorem
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Compared to basic modal logic

- there are no sweeping general results such as Sahlqvist’s theorem
- there is no comprehensive completeness theory (duality, canonicity, filtration, . . .)
Overview

- Introduction
- Obstacles
- Completeness for $\mu$ML
- Completeness for flat fixpoint logics
- Frame conditions
- Conclusions
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Obstacle 1: computational danger zone

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- Language: $\diamond_R, \diamond_U$
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- Language: $\diamond_R, \diamond_U$
- Intended Semantics: $\mathbb{N} \times \mathbb{N}$
  - $(m, n)R(m', n')$ iff $m' = m + 1$ and $n' = n$
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- **Logic** $K_G := K +$
  - **functionality**: $\Diamond_R p \leftrightarrow \Box_R p$ and $\Diamond_U p \leftrightarrow \Box_U p$
  - **confluence**: $\Diamond_R \Box_U p \rightarrow \Box_U \Diamond_R p$

$K_G$ is sound and complete with respect to its Kripke frames.

Add master modality, $\langle \ast \rangle p := \mu x. p \lor \Diamond_R x \lor \Diamond_U x$

$\mu K_G$ is sound but incomplete with respect to its Kripke frames.

Proof: Use recurrent tiling problem to show that

the $\Diamond_R, \Diamond_U, \langle \ast \rangle$-logic of $Fr(K_G)$ is not recursively enumerable.
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  - Proof: Use recurrent tiling problem to show that
  - the $\Diamond_R, \Diamond_U, \langle \ast \rangle$-logic of $Fr(KG)$ is not recursively enumerable
Example: \( \langle * \rangle p := \bigvee_{n \in \omega} \Box^n p \)

\( \{\langle * \rangle p\} \cup \{\Box^n \neg p \mid n \in \omega\} \) is finitely satisfiable but not satisfiable
Obstacle 2: compactness failure

▶ Example: \( \langle * \rangle p := \bigvee_{n \in \omega} \diamond^n p \)
  
  ▶ \( \{ \langle * \rangle p \} \cup \{ \square^n \neg p \mid n \in \omega \} \) is finitely satisfiable but not satisfiable

▶ Fixpoint logics have no nice Stone-based duality
Obstacle 3: fixpoint alternation

- tableaux: fixpoint unfolding
  - $\nu$-fixpoints may be unfolded infinitely often
  - $\mu$-fixpoints may only be unfolded finitely often
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- tableaux: fixpoint unfolding
  - $\nu$-fixpoints may be unfolded infinitely often
  - $\mu$-fixpoints may only be unfolded finitely often
- with every branch of tableau associate a trace graph
- obstacle 3a: conjunctions cause trace proliferation
- obstacle 3b: fixpoint alternations cause intricate combinatorics
What to do?

▶ consider simple frame conditions only (if at all)
▶ restrict language to fixpoints of simple formulas (avoid alternation)
▶ allow alternation, but develop suitable combinatorial framework
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Completeness

Kozen Axiomatisation:

- complete calculus for modal logic
- $\varphi(\mu p. \varphi) \vdash_{K} \mu p. \varphi$
- if $\varphi(\psi) \vdash_{K} \varphi$ then $\mu p. \varphi \vdash_{K} \psi$

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**Theorem** (Kozen 1983)

$\vdash_K$ is sound, and complete for a conjunctive formulas.
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Kozen Axiomatisation:
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**Theorem** (Kozen 1983)
$\vdash_{\mathcal{K}}$ is sound, and complete for aconjunctive formulas.

**Theorem** (Walukiewicz 1995)
$\vdash_{\mathcal{K}}$ is sound and complete for all formulas.
Our Aim

- understand general principles underlying completeness for $\mu$ML
- integrate Kozen-Walukiewicz Theorem in theory of modal logic
- generalise completeness theorem to wider setting
Walukiewicz’ Proof: Evaluation

Why is Walukiewicz’ proof hard?
Walukiewicz’ Proof: Evaluation

Why is Walukiewicz’ proof hard?

1. complex combinatorics of traces
2. incorporate simulation theorem into derivations
3. mix of \( \vdash_k \)-derivations, tableaux and automata
4. tableau rules for boolean connectives complicate combinatorics
5. . . .
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content vs wrapping
Our Approach: Principles

- separate the combinatorics from the dynamics
- focus on automata rather than formulas
- make traces first-class citizens
Our Approach: Principles

**Dynamics**: coalgebra
- one step at a time
- absorb booleans into one-step rules

**Combinatorics**: trace management
- use binary relations to deal with trace combinatorics

**Automata**
- uniform, 'clean' presentation of fixpoint formulas
- excellent framework for developing trace theory
- direct formulation of simulation theorem
- bring automata into proof theory
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Theorem (Enqvist, Seifan & YV)
There are maps $\mathbb{B}_- : \mu\text{ML} \rightarrow \text{Aut}(\text{ML}_1)$ and $\xi : \text{Aut}(\text{ML}_1) \rightarrow \mu\text{ML}$ that
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As a corollary, we may apply proof-theoretic concepts to automata
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1. preserve meaning: $\varphi \equiv \mathcal{B}_\varphi$ and $A \equiv \xi(A)$
2. satisfy $\varphi \equiv_K \xi(\mathcal{B}_\varphi)$;
3. interact nicely with Booleans, modalities, fixpoints, and substitution:

$$\xi(A[\mathcal{B}/x]) \equiv_K \xi(A)[\xi(\mathcal{B})/x].$$
Automata & Formulas

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Games for Automata

Satisfiability Game $S(A)$ (Fontaine, Leal & YV 2010)

- basic positions: binary relations $R \in P(A \times A)$
- $R$ corresponds to $\bigwedge \{\Theta(a) \mid a \in R\}$
- direct representation of $A$-traces through $R_0 R_1 \cdots$
- $\exists$ wins $S(A)$ iff $L(A) \neq \emptyset$
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Consequence Game $C(\mathbb{A}, \mathbb{A}')$
- basic positions: pair of binary relations $(R, R')$
- winning condition in terms of trace reflection
- $\mathbb{A} \models_G \mathbb{A}'$ indicates a tight structural link between $\mathbb{A}$ and $\mathbb{A}'$
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Special Automata

Modal Automaton: $\mathbb{A} = \langle A, \Theta, \Omega, a_I \rangle$, with $\Theta : A \to \text{ML}_1(P, A)$

- $\text{Latt}(A) \pi ::= p | \pi \lor \pi | \bot | \pi \land \pi | \top$
- $\text{ML}_1(P, A) \alpha ::= p | \neg p | \Diamond \alpha | \Box \alpha | \alpha \lor \alpha | \bot | \alpha \land \alpha | \top$
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Disjunctive Automaton \( \Theta : A \to ML_1^d(P, A) \)

- \( List(P) \pi ::= \bot \mid \top \mid p \land \pi \mid \neg p \land \pi \)
- \( ML_1^d(P, A) \alpha ::= \bot \mid \top \mid \pi \land \nabla B \mid \alpha \lor \alpha, \)

where \( B \subseteq A \) and \( \nabla B ::= \bigwedge \diamond B \land \square \lor B. \)
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Disjunctive Automaton $\Theta : A \rightarrow \text{ML}_1^d(P, A)$
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Semi-disjunctive Automaton $\Theta(a) \in \text{ML}_1^{s,C}(P, A)$
- $\text{List}(P)\!$ $\pi ::= \bot \mid \top \mid p \land \pi \mid \neg p \land \pi$
- $\text{ML}_1^{s,C}(P, A)\!$ $\alpha ::= \bot \mid \top \mid \pi \land \nabla \{ \bigwedge B \mid B \in B \} \mid \alpha \lor \alpha$, where for all $B \in B$, all $b, b' \in B$ with $b \neq b'$, $b$ or $b'$ is a maximal even element of $C$. 
Key Lemmas

**Strong Simulation Theorem** (cf W39)
For every modal automaton $\mathcal{A}$ there is an equivalent disjunctive simulation $\overline{\mathcal{A}}$ such that

$$
\mathcal{A} \models_G \overline{\mathcal{A}} \\
\overline{\mathcal{A}} \models_G \mathcal{A} \\
\mathcal{B}[\overline{\mathcal{A}}/x] \models_G \mathcal{B}[\mathcal{A}/x]
$$

for all automata $\mathcal{B}$.

**Lemma** (cf W36)
Let $\mathcal{A}, \mathcal{B}$ be respectively a semidisjunctive and an arbitrary automaton. If $\mathcal{A} \models_G \mathcal{B}$, then $\mathcal{A} \land \neg \mathcal{B}$ has a thin refutation.

**Lemma** (cf Kozen)
If $\mathcal{A}$ is a consistent automaton, then $\exists$ has a winning strategy in $S_{\text{thin}}$.

**Corollary** If $\mathcal{A}$ is a consistent (semi-)disjunctive automaton, then $\mathcal{A}$ is satisfiable.
Main Proposition
For every $\varphi \in \mu\text{ML}$ there is an equivalent disjunctive automaton $D$ such that

$$\varphi \vdash_K D.$$
Proof of Kozen-Walukiewicz Theorem

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Proof
Induction on $\varphi$
(similar to Walukiewicz’ proof, but using the above lemmas.)
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Completeness for $\mu ML$ is almost immediate from this.
Theorem (Enqvist, Seifan & YV)
Assume that

- \( \mathcal{L} \) is a one-step language with \textit{an adequate disjunctive base}
- \( \mathcal{H} \) is a one-step sound and complete axiomatization for \( \mathcal{L} \)

Then \( \mathcal{H} + \text{Koz} \) is a sound and complete axiomatization for \( \mu\mathcal{L} \).
**Theorem** (Enqvist, Seifan & YV)
Assume that

- $\mathcal{L}$ is a one-step language with an adequate disjunctive base
- $\mathcal{H}$ is a one-step sound and complete axiomatization for $\mathcal{L}$

Then $\mathcal{H} + Koz$ is a sound and complete axiomatization for $\mu\mathcal{L}$.

Examples:

- linear time $\mu$-calculus
- $k$-successor $\mu$-calculus
- standard modal $\mu$-calculus
- graded $\mu$-calculus
- monotone modal $\mu$-calculus
- game $\mu$-calculus
- ...
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Fix a basic modal formula $\gamma(x, \bar{p})$, positive in $x$
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Add a fixpoint connective $\sharp_{\gamma}$ to the language of ML
(arity of $\sharp_{\gamma}$ depends on $\gamma$ but notation hides this)
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Example: $Upq := \mu x. p \lor (q \land \Box x)$,

now: $Upq := \sharp_\gamma(p, q)$ with $\gamma = p \lor (q \land \Box x)$

Intended reading: $\sharp_\gamma(\bar{\varphi}) \equiv \mu x. \gamma(x, \bar{\varphi})$ for any $\bar{\varphi} = (\varphi_1, \ldots, \varphi_n)$. 
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Obtain language $ML_\gamma$:

$$\varphi ::= p \mid \neg p \mid \bot \mid T \mid \varphi_1 \lor \varphi_2 \mid \varphi_1 \land \varphi_2 \mid \Diamond_i \varphi \mid \Box_i \varphi \mid \sharp_\gamma(\vec{\varphi})$$
Flat Modal Fixpoint Logics: Syntax

- Fix a basic modal formula $\gamma(x, \bar{p})$, positive in $x$
- Add a fixpoint connective $\#_\gamma$ to the language of ML
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- Example: $Upq := \mu x. p \lor (q \land \Box x)$,
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$$\varphi ::= p \mid \neg p \mid \bot \mid T \mid \varphi_1 \lor \varphi_2 \mid \varphi_1 \land \varphi_2 \mid \Diamond i\varphi \mid \Box i\varphi \mid \#_\gamma(\bar{\varphi})$$

- Examples: CTL, LTL, (PDL), common knowledge, ATL, …
Flat Modal Fixpoint Logics: Kripke Semantics

- Kripke frame $S = \langle S, R \rangle$ with $R \subseteq S \times S$.
- Complex algebra: $S^+ := \langle \wp(S), \emptyset, S, \sim_S, \cup, \cap, \langle R \rangle \rangle$, 
  $\langle R \rangle : \wp(S) \rightarrow \wp(S)$ given by 
  $\langle R \rangle(X) := \{ s \in S \mid Rst \text{ for some } t \in X \}$
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  $\langle R \rangle: \wp(S) \to \wp(S)$ given by 
  $\langle R \rangle(X) := \{s \in S \mid Rst \text{ for some } t \in X\}$
- Every modal formula $\varphi(p_1, \ldots, p_n)$ corresponds to a term function 
  $\varphi^S: \wp(S)^n \to \wp(S)$.
- $\gamma$ positive in $x$, hence $\gamma^S$ order preserving in $x$. 
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$\gamma$ positive in $x$, hence $\gamma^S$ order preserving in $x$.

By Knaster-Tarski we may define $\#^S : \wp(S)^n \to \wp(S)$ by
$\#^S(\vec{B}) := \text{LFP.}\gamma^S(-, \vec{B})$. 
Flat Modal Fixpoint Logics: Kripke Semantics

- Kripke frame $S = \langle S, R \rangle$ with $R \subseteq S \times S$.
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  $\#^S(\vec{B}) := \text{LFP}.\gamma^S(\vec{-}, \vec{B})$.
- Kripke $\#$-algebra $S^\# := \langle \wp(S), \emptyset, S, \sim_S, \cup, \cap, \langle R \rangle, \#^S \rangle$. 
Candidate Axiomatization

\[ \mathbb{K}_\gamma := \mathbb{K} \text{ extended with} \]

- **prefixpoint axiom:**
  \[ \gamma(\#(\varphi), \varphi) \vdash \#(\varphi) \]

- **Park’s induction rule:**
  from \( \gamma(\psi, \varphi) \vdash \psi \) infer \( \#_\gamma(\varphi) \vdash \psi \).
Flat Modal Fixpoint Logics: Algebraic completeness proof

$\text{A} = \langle \text{A}, \perp, \top, \neg, \land, \lor, \exists, \#$ \rangle$ with $\# : \text{A} \rightarrow \text{A}$ satisfying $\#(\vec{b}) = \text{LFP}_{\vec{b}}$, where $\gamma_{\text{A}}(\vec{b}) : \text{A} \rightarrow \text{A}$ is given by $\gamma_{\text{A}}(\vec{b})(a) = \gamma_{\text{A}}(a, \vec{b})$.

Axiomatically: modal $\#$-algebras satisfy

$\gamma(\#(\vec{y}), \vec{y}) \leq \#(\vec{y})$

if $\gamma(x, \vec{y}) \leq x$ then $\#(\vec{y}) \leq x$.

Completeness for flat fixpoint logics: $\text{Equ}(\text{MA} \#) = \text{Equ}(\text{KA} \#)$

Two key concepts:

- constructiveness
- $O$-adjointness
Modal $\#$-algebra: $A = \langle A, \perp, \top, \neg, \land, \lor, \diamond, \# \rangle$ with $\#: A^n \to A$ satisfying
\[
\#(\vec{b}) = \text{LFP}.\gamma^A_{\vec{b}},
\]
where $\gamma^A_{\vec{b}} : A \to A$ is given by $\gamma^A_{\vec{b}}(a) := \gamma^A(a, \vec{b})$. 
Flat Modal Fixpoint Logics: Algebraic completeness proof

- **Modal $\#$-algebra:** $A = \langle A, \bot, \top, \neg, \land, \lor, \Diamond, \# \rangle$ with $\# : A^n \to A$
satisfying

$$\#(\vec{b}) = \text{LFP}.\gamma^A_{\vec{b}},$$

where $\gamma^A_{\vec{b}} : A \to A$ is given by $\gamma^A_{\vec{b}}(a) := \gamma^A(a, \vec{b}).$

- **Axiomatically:** modal $\#$-algebras satisfy
  - $\gamma(\#(\vec{y}), \vec{y}) \leq \#(\vec{y})$
  - if $\gamma(x, \vec{y}) \leq x$ then $\#(\vec{y}) \leq x.$

- **Completeness for flat fixpoint logics:** $\text{Equ}(\text{MA}_\#) \equiv \text{Equ}(\text{KA}_\#)$
Flat Modal Fixpoint Logics: Algebraic completeness proof

- **Modal ♯-algebra:** $A = \langle A, \bot, \top, \neg, \land, \lor, \Diamond, \# \rangle$ with $\# : A^n \rightarrow A$
  satisfying

  $$\#(\vec{b}) = \text{LFP} \cdot \gamma^A_{\vec{b}},$$

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- **Axiomatically:** modal ♯-algebras satisfy
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- **Completeness for flat fixpoint logics:** $\text{Equ}(\text{MA}_\#) \equiv \text{Equ}(\text{KA}_\#)$

- **Two key concepts:**
  - constructiveness
  - $\mathcal{O}$-adjointness
An $\text{MA}_\#_\#$-algebra $\mathcal{A}$ is constructive if

$$\#(\vec{b}) = \bigvee_{n \in \omega} \gamma^n_b(\bot).$$
Constructiveness

An MA_♯-algebra A is constructive if

$$\#(\vec{b}) = \bigvee_{n \in \omega} \gamma^n_b(\bot).$$

Note: we do not require A to be complete!
Constructiveness

- An MA_{♯}-algebra △ is **constructive** if

\[ \#(\vec{b}) = \bigvee_{n \in \omega} \gamma^n_b(\bot). \]

Note: we do not require △ to be complete!

**Theorem** (Santocanale & Venema)
Let A be a countable, residuated, modal ♯-algebra.
If A is constructive, then A can be embedded in a Kripke ♯-algebra.
Constructiveness

An $\text{MA}_\#$-algebra $\mathbb{A}$ is constructive if

$$\#(\vec{b}) = \bigvee_{n \in \omega} \gamma^n_b(\bot).$$

Note: we do not require $\mathbb{A}$ to be complete!

**Theorem** (Santocanale & Venema)

Let $A$ be a countable, residuated, modal $\#$-algebra.
If $A$ is constructive, then $A$ can be embedded in a Kripke $\#$-algebra.

**Proof**

Let $f : (P, \leq) \rightarrow (Q, \leq)$ be an order-preserving map.
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- $f$ is a (left) adjoint or residuated if it has a residual $g : Q \rightarrow P$ with

$$fp \leq q \iff p \leq gq.$$
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- $f$ is a (left) $\mathcal{O}$-adjoint if it has an $\mathcal{O}$-residual $G_f : Q \rightarrow \wp(\mathcal{O}P)$ with
  \[ fp \leq q \iff p \leq y \text{ for some } y \in G_f q. \]
\(\mathcal{O}\)-adjoints

Let \(f : (P, \leq) \rightarrow (Q, \leq)\) be an order-preserving map.

- \(f\) is a (left) adjoint or residuated if it has a residual \(g : Q \rightarrow P\) with
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- \(f\) is a (left) \(\mathcal{O}\)-adjoint if it has an \(\mathcal{O}\)-residual \(G_f : Q \rightarrow \wp(\omega(P))\) with
  \[fp \leq q \iff p \leq y\] for some \(y \in G_f q\).

**Proposition** (Santocanale 2005)

- \(f\) is a left adjoint iff \(f\) is a join-preserving \(\mathcal{O}\)-adjoint
**$\mathcal{O}$-adjoints**

Let $f : (P, \leq) \rightarrow (Q, \leq)$ be an order-preserving map.

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**Proposition** (Santocanale 2005)

- $f$ is a left adjoint iff $f$ is a join-preserving $\mathcal{O}$-adjoint
- $\mathcal{O}$-adjoints are Scott continuous
Let \( f : (P, \leq) \to (Q, \leq) \) be an order-preserving map.

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  \]

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**Proposition** (Santocanale 2005)

- \( f \) is a left adjoint iff \( f \) is a join-preserving \( \mathcal{O} \)-adjoint
- \( \mathcal{O} \)-adjoints are Scott continuous
- \( \land \) is continuous but not an \( \mathcal{O} \)-adjoint.
Finitary $\mathcal{O}$-adjoints

Let $f : A^n \to A$ be an $\mathcal{O}$-adjoint with $\mathcal{O}$-residual $G$. 

Theorem (Santocanale 2005)
If $f : A^n \to A$ is a finitary $\mathcal{O}$-adjoint, then $\text{LFP}_f$, if existing, is constructive.
Finitary $\mathcal{O}$-adjoints

Let $f : A^n \to A$ be an $\mathcal{O}$-adjoint with $\mathcal{O}$-residual $G$.

- Inductively define $G^n : A \to \wp(A)$

\[
\begin{align*}
G^0(a) & := \{a\} \\
G^{n+1}(a) & := G[G^n(a)]
\end{align*}
\]
Finitary $\mathcal{O}$-adjoints

Let $f : A^n \rightarrow A$ be an $\mathcal{O}$-adjoint with $\mathcal{O}$-residual $G$.

- Inductively define $G^n : A \rightarrow \wp(\wp^n)$:
  
  \[
  G^0(a) := \{a\} \quad G^{n+1}(a) := G[G^n(a)]
  \]

- Call $f$ finitary if $G^\omega(a) := \bigcup_{n \in \omega} G^n(a)$ is finite.
Finitary $\mathcal{O}$-adjoints

Let $f : A^n \to A$ be an $\mathcal{O}$-adjoint with $\mathcal{O}$-residual $G$.

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If $f : A \to A$ is a finitary $\mathcal{O}$-adjoint, then $\text{LFP}.f$, if existing, is constructive.
Adjoints on free algebras

Free modal ($\#$-) algebras have many $\Omega$-adjoints!

cf. free distributive lattice are Heyting algebras,

Whitman's rule for free lattices, . . .

Call a modal formula $\gamma$ untied in $x$ if it belongs to

$\gamma ::= x \mid \top \mid \gamma \lor \gamma \mid \psi \land \gamma \mid \nabla\{\gamma_1, \ldots, \gamma_n\}$

where $\psi$ does not contain $x$

Examples:

$3x, 2x, 3x \land 33x \land 2p, 3x \land 32x \land 2(3x \lor 32x)$, . . .

Counterexamples:

$3(x \land 3x), 3x \land 23x$

Theorem

(Santocanale & YV 2010)

Untied formulas are finitary $\Omega$-adjoints.
Free modal ($\Diamond$-)algebras have many $\mathcal{O}$-adjoints!

Examples:
- $3x$, $2x$, $3x \land 33x \land 2p$, $3x \land 32x \land 2(3x \lor 32x)$, . . .

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Free modal ($\mathcal{O}$-)algebras have many $\mathcal{O}$-adjoints!

- cf. free distributive lattice are Heyting algebras,
- Examples:
  - $3x$, $2x$, $3x \land 33x \land 2p$, $3x \land 32x \land 2(3x \lor 32x)$, ...
- Counterexamples:
  - $3(x \land 3x)$, $3x \land 23x$.

Theorem (Santocanale & YV 2010)
Untied formulas are finitary $\mathcal{O}$-adjoints.
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Free modal (♯-)algebras have many $O$-adjoints!

- cf. free distributive lattice are Heyting algebras,
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Call a modal formula $\gamma$ untied in $x$ if it belongs to

$$\gamma ::= x \mid T \mid \gamma \lor \gamma \mid \psi \land \gamma \mid \nabla\{\gamma_1, \ldots, \gamma_n\}$$

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Free modal (\(\#\))-algebras have many \(O\)-adjoints!

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where \(\psi\) does not contain \(x\)

- Examples: \(\Diamond x\), \(\Box x\), \(\Diamond x \land \Diamond \Diamond x \land \Box p\), \(\Diamond x \land \Diamond \Box x \land \Box(\Diamond x \lor \Diamond \Box x)\), . . .
Free modal (♯-)algebras have many $O$-adjoints!
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- Examples: $\Diamond x$, $\Box x$, $\Diamond x \land \Diamond \Diamond x \land \Box p$, $\Diamond x \land \Diamond \Box x \land \Box (\Diamond x \lor \Diamond \Box x)$, . . .
- Counterexamples: $\Diamond (x \land \Diamond x)$, $\Diamond x \land \Box \Diamond x$
Free modal ($\mathcal{H}$-)algebras have many $\mathcal{O}$-adjoints!

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**Theorem** (Santocanale & YV 2010)

Untied formulas are finitary $\mathcal{O}$-adjoints.
A general result

Call a modal formula $\gamma$ untied in $x$ if it belongs to

$$\gamma ::= x \mid \top \mid \gamma \lor \gamma \mid \psi \land \gamma \mid \nabla\{\gamma_1, \ldots, \gamma_n\}$$

where $\psi$ does not contain $x$.
A general result

- Call a modal formula $\gamma$ **untied in $x$** if it belongs to

$$\gamma ::= x \mid T \mid \gamma \lor \gamma \mid \psi \land \gamma \mid \nabla\{\gamma_1, \ldots, \gamma_n\}$$

where $\psi$ does not contain $x$

- Examples: $\diamond x$, $\square x$, $\diamond x \land \diamond \diamond x \land \square p$, $\diamond x \land \diamond \square x \land \square (\diamond x \lor \diamond \square x)$, …
A general result

Call a modal formula $\gamma$ untied in $x$ if it belongs to

$$\gamma ::= x \mid T \mid \gamma \lor \gamma \mid \psi \land \gamma \mid \nabla\{\gamma_1, \ldots, \gamma_n\}$$

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- Examples: $\Diamond x$, $\Box x$, $\Diamond x \land \Box \Box x \land \Box p$, $\Diamond x \land \Box \Box x \land \Box(\Diamond x \lor \Diamond \Box x)$, $\ldots$
- Non-examples: $\Diamond(x \land \Diamond x)$, $\Diamond x \land \Box \Diamond x$
A general result

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- Examples: $\Box x$, $\Box x$, $\Box x \land \Box x \land \Box p$, $\Box x \land \Box x \land \Box (\Box x \lor \Box x)$, ...

- Non-examples: $\Box (x \land \Box x)$, $\Box x \land \Box x$

**Theorem** (Santocanale & YV 2010)

Let $\gamma$ be untied wrt $x$. Then $K_\gamma$ is sound and complete wrt its Kripke semantics.
A general result

▶ Call a modal formula $\gamma$ untied in $x$ if it belongs to

$$\gamma ::= x \mid T \mid \gamma \lor \gamma \mid \psi \land \gamma \mid \nabla\{\gamma_1, \ldots, \gamma_n\}$$

where $\psi$ does not contain $x$

▶ Examples: $\diamond x$, $\square x$, $\diamond x \land \diamond \diamond x \land \square p$, $\diamond x \land \diamond \square x \land \square (\diamond x \lor \diamond \square x)$, $\ldots$

▶ Non-examples: $\diamond (x \land \diamond x)$, $\diamond x \land \square \diamond x$

**Theorem** (Santocanale & YV 2010)

Let $\gamma$ be untied wrt $x$. Then $K_\gamma$ is sound and complete wrt its Kripke semantics.

**Notes**
A general result

Call a modal formula $\gamma$ untied in $x$ if it belongs to

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- Non-examples: $\Diamond (x \land \Diamond x)$, $\Diamond x \land \Box \Diamond x$

**Theorem** (Santocanale & YV 2010)
Let $\gamma$ be untied wrt $x$. Then $K_\gamma$ is sound and complete wrt its Kripke semantics.

Notes
- Santocanale & YV have fully general result for extended axiom system.
A general result

Call a modal formula $\gamma$ untied in $x$ if it belongs to

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- Non-examples: $\diamond (x \land \diamond x)$, $\diamond x \land \Box \diamond x$

**Theorem** (Santocanale & YV 2010)
Let $\gamma$ be untied wrt $x$. Then $K_\gamma$ is sound and complete wrt its Kripke semantics.

**Notes**
- Santocanale & YV have fully general result for extended axiom system.
- Schröder & YV have similar results for wider coalgebraic setting.
Overview

- Introduction
- Obstacles
- Completeness for $\mu$ML
- Completeness for flat fixpoint logics
- Frame conditions
- Conclusions
**Conjecture** Let $L$ be an extension of $K_\Gamma$ or $K_\mu$ with an axiom set $\Phi$ such that each $\varphi \in \Phi$

- is canonical
- corresponds to a *universal* first-order frame condition.

Then $L$ is sound and complete for the class of frames satisfying $\Phi$. 
Overview

- Introduction
- Obstacles
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- framework for proving completeness for $\mu$-calculi
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- framework for proving completeness for $\mu$-calculi
- perspective for bringing automata into proof theory
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  - general completeness result for coalgebraic $\mu$-calculi
Conclusions

- framework for proving completeness for $\mu$-calculi
  - perspective for bringing automata into proof theory
  - general completeness result for coalgebraic $\mu$-calculi
- general completeness result for flat fixpoint logics
Future work

- prove conjecture on frame conditions!
- prove completeness for fragments of $\mu$ML (game logic!)
  - many $\mu$ML-fragments have automata-theoretic counterparts!
- interpolation for fixpoint logics (PDL!)
- fixpoint logics on non-boolean basis
  - non-boolean automata?
- proof theory for modal automata
- further explore notion of $\mathcal{O}$-adjointness
- ...
References

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http://staff.science.uva.nl/~yde
THANK YOU!